Generalized Optimum Receivers of Gaussian Signals

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Optimum reception of two zero-mean Gaussian signals is accomplished by comparing a quadratic form $\iint x(s)H(s,t)x(t)\,ds\,dt$ *in the observable waveform* $x(t)$ *with a predetermined threshold, if the symmetric kernel* $H(s,t)$ *can be given as a square-integrable solution of*

$$\iint R_1(s,u)H(u,v)R_2(v,t)\,du\,dv = R_2(s,t) - R_1(s,t),$$

where $R_1(s,t)$ *and* $R_2(s,t)$ *are the covariances of the two signals. In this paper, we generalize this result so that* $\sum_{l,m} \iint x^{(l)}(s)H_{lm}(s,t)x^{(m)}(t)\,ds\,dt$ *is the quadratic form to be used and* $\{H_{lm}(s,t)\}$ *is given as a formal solution of*

$$\sum_{l,m} \iint \frac{\partial^l}{\partial u^l} R_1(s,u)H_{lm}(u,v)\frac{\partial^m}{\partial v^m} R_2(v,t)\,du\,dv = R_2(s,t) - R_1(s,t).$$

In other words, the generalized quadratic form is in the derivatives of $x(t)$ *as well as* $x(t)$ *itself and the kernels* $H_{lm}(s,t)$ *consist of two-dimensional* δ*-functions in addition to square-integrable functions. This result is extended to the case of two nonzero-mean signals and then to the case of* M *Gaussian signals in noise.*

I. INTRODUCTION

Consider the problem of discriminating between two zero-mean Gaussian signals by observing the sample function $x(t)$, $0 \leqq t \leqq 1$. We assume that their covariances $R_1(s,t)$ and $R_2(s,t)$ are continuous and positive-definite on $[0,1] \times [0,1]$. According to previous results,[1,2,3] if the integral equation

$$\int_0^1 \int_0^1 R_1(s,u)H(u,v)R_2(v,t)\,du\,dv = R_2(s,t) - R_1(s,t), \quad 0 \leqq s,t \leqq 1, \tag{1}$$

has a symmetric and square-integrable solution $H(s,t)$, then the following decision scheme minimizes the error probability:

$$\text{choose} \quad R_1(s,t) \quad \text{if} \quad \int_0^1 \int_0^1 x(s)H(s,t)x(t)\,ds\,dt < c, \tag{2}$$

$$\text{choose} \quad R_2(s,t) \text{ otherwise,}$$

where

$$c = 2 \log \frac{\alpha_1}{\alpha_2} - \sum_{i=0}^{\infty} \log \lambda_i \ , \tag{3}$$

in which α_1 and α_2 are the *a priori* probabilities associated with the two signals, and $\lambda_i > 0$, $i = 0, 1, 2, \cdots$, are the eigenvalues of an operator $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$.*

Unfortunately, existence of a square-integrable solution of (1) is too restrictive a condition. Thus, relaxation of the condition, which amounts to generalization of the quadratic form of (2), is desirable. In this paper, we accomplish this in two ways: one is to allow $H(s,t)$ to contain δ-functions as well as square-integrable functions, resulting in the generalization of the structure of the quadratic form; the other is to consider the derivatives of $x(t)$ as well as $x(t)$ itself, thus generalizing the elements of the quadratic form. The result is extended to the case where the means of the two signals are nonzero, and is further extended to the case of M Gaussian signals in noise.

II. GENERALIZED OPTIMUM RECEIVER OF TWO ZERO-MEAN GAUSSIAN SIGNALS

Consider the following generalization of the quadratic form of (2):

$$Q(x) = \sum_{l,m=0}^{r} \int_0^1 \int_0^1 x^{(l)}(s) H_{lm}(s,t) x^{(m)}(t)\,ds\,dt, \tag{4}$$

where $x^{(l)}(t)$ is the lth derivative of $x(t)$, and

$$H_{lm}(s,t) = \sum_{j,k=1}^{n_1} a_{jklm}\,\delta(s - s_j)\,\delta(t - s_k)$$

* More precisely, λ_i, $i = 0, 1, 2, \ldots$, are the eigenvalues of the extension of $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$ to the whole of $\mathfrak{L}_2$, where R_1 and R_2 denote the integral operators with the kernels $R_1(s,t)$ and $R_2(s,t)$, and $\mathfrak{L}_2$ the space of all square-integrable functions on [0,1]. We recall that existence of a symmetric, square-integrable solution of (1) implies that $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$ has a unique bounded extension to the whole of $\mathfrak{L}_2$ having eigenvalues $\{\lambda_i\}$ such that $0 < \prod_{i=0}^{\infty} \lambda_i < \infty$.

$$+ \sum_{j=1}^{n_2} [\delta(s - t_j)h_{jlm}(t) + \tilde{h}_{jlm}(s)\ \delta(t - t_j)]$$

$$+ \hat{h}_{lm}(s)\ \delta(s - t) + \tilde{H}_{lm}(s,t), \tag{5}$$

in which a_{jklm} are real constants, and $0 \leqq s_j\ , t_j \leqq 1$, and $h_{jlm}(t)$, $h_{jlm}(t)$ and $\hat{h}_{lm}(t)$ are square-integrable functions on [0, 1] while $\tilde{H}_{lm}(s, t)$ are square-integrable functions on [0, 1] $\times$ [0, 1]. In writing (4), we have assumed that almost all sample functions of both signals have rth derivatives.* Note that the nonsquare-integrable part of $H_{lm}(s,t)$ consists of three types of two-dimensional δ-functions: (*i*) those at points and their mirror images with respect to the diagonal $s = t$, (*ii*) those along horizontal lines (t = constant) and their mirror images (s = constant), and (*iii*) those along the diagonal. By formally substituting (5) into (4), we obtain an explicit form of $Q(x)$, namely,

$$Q(x) = \sum_{l,m=0}^{r} \Bigg[\sum_{j,k=1}^{n_1} a_{jklm} x^{(l)}(s_j) x^{(m)}(s_k)$$

$$+ \sum_{j=1}^{n_2} x^{(l)}(t_j) \int_0^1 [h_{jlm}(t) + \tilde{h}_{jlm}(t)] x^{(m)}(t)\ dt$$

$$+ \int_0^1 x^{(l)}(t) \hat{h}_{lm}(t) x^{(m)}(t)\ dt + \int_0^1 \int_0^1 x^{(l)}(s) \tilde{H}_{lm}(s,t) x^{(m)}(t)\ ds\ dt \Bigg]. \tag{6}$$

As the corresponding generalization of the integral equation (1), we consider the following:

$$\sum_{l,m=0}^{r} \int_0^1 \int_0^1 \frac{\partial^l}{\partial u^l} R_1(s,u) H_{lm}(u,v) \frac{\partial^m}{\partial v^m} R_2(v,t)\ du\ dv$$

$$= R_2(s,t) - R_1(s,t), \qquad 0 \leqq s,t \leqq 1. \tag{7}$$

Again, through formal substitution of (5), (7) becomes

$$\sum_{l,m=0}^{r} \Bigg\{ \sum_{j,k=1}^{n_1} a_{jklm} \frac{\partial^l}{\partial t^l} R_1(s,t)|_{t=s_j} \frac{\partial^m}{\partial u^m} R_2(s,t)|_{s=s_k} + \sum_{j=1}^{n_2} \int_0^1$$

$$\cdot \Bigg[\frac{\partial^l}{\partial t^l} R_1(s,t)|_{t=t_j} h_{jlm}(u) \frac{\partial^m}{\partial u^m} R_2(u,t)$$

$$+ \frac{\partial^l}{\partial u^l} R_1(s,u) \tilde{h}_{jlm}(u) \frac{\partial^m}{\partial s^m} R_2(s,t)|_{s=t_j} \Bigg] du$$

* A simple sufficient condition for this is existence of $(\partial^{2r+2}/\partial s^{r+1}\partial t^{r+1})\ R_i(s,t)$, $i = 1, 2$.[4]

$$+ \int_0^1 \frac{\partial^l}{\partial u^l} R_1(s,u)\hat{h}_{lm}(u) \frac{\partial^m}{\partial u^m} R_2(u,t)\, du + \int_0^1 \int_0^1 \frac{\partial^l}{\partial u^l} R_1(s,u)\tilde{H}_{lm}(u,v)$$

$$\left. \cdot \frac{\partial^m}{\partial v^m} R_2(v,t)\, du\, dv \right\} = R_2(s,t) - R_1(s,t), \qquad 0 \leqq s,t \leqq 1, \tag{8}$$

where we have assumed that $(\partial^{2r}/\partial s^r \partial t^r)R_1(s,t)$ and $(\partial^{2r}/\partial s^r \partial t^r)R_2(s,t)$ exist and are continuous on $[0,1] \times [0,1]$.

Unlike $H(s,t)$ of (2), which is uniquely given as the symmetric, square-integrable solution of (1),* the defining elements of $Q(x)$ (i.e., $\{a_{jklm}\}$, $\{s_j\}$, $\{t_j\}$, $\{h_{jlm}(t)\}$, $\{\tilde{h}_{jlm}(t)\}$, $\{\hat{h}_{lm}(t)\}$, $\{H_{lm}(s,t)\}$) cannot be uniquely determined by (8) in general for a given pair of covariances $R_1(s,t)$ and $R_2(s,t)$. Nevertheless, we can establish the following:

If (*i*) $R_1(s,t)$ and $R_2(s,t)$ are positive-definite,
(*ii*) $(\partial^{2r}/\partial s^r \partial t^r)R_1(s,t)$ and $(\partial^{2r}/\partial s^r \partial t^r)R_2(s,t)$ are continuous,
(*iii*) for almost all sample functions both signals have rth derivatives,† and
(*iv*) there exist some set of finite sequences $\{a_{jklm}\}$, $\{s_j\}$, $\{t_j\}$, $\{h_{jlm}(t)\}$, $\{\tilde{h}_{jlm}(t)\}$, $\hat{h}_{lm}(t)\}$ and $\{\tilde{H}_{lm}(s,t)\}$ which satisfy (8), then the decision scheme (2) with $\int_0^1\int_0^1 x(s)H(s,t)x(t)dsdt$ replaced by $Q(x)$ of (6) is optimum.

The proof is based on two measure theoretical facts: (*i*) two probability measures P_1 and P_2 corresponding to two Gaussian signals are either equivalent or singular,‡[5,6] and (*ii*) if they are equivalent then there is a special random variable called the Radon-Nikodym derivative $(dP_2/dP_1)(x)$, in terms of which the optimum decision scheme is specified as follows:[1]

$$\text{choose} \quad R_1(s,t) \quad \text{if} \quad \frac{dP_2}{dP_1}(x) < \frac{\alpha_1}{\alpha_2},$$

$$\text{choose} \quad R_2(s,t) \quad \text{otherwise.}$$

Hence, in the Appendix, we first prove that existence of $\{a_{jklm}\}$, $\{s_j\}$, $\{t_j\}$, $\{h_{jlm}(t)\}$, $\{\tilde{h}_{jlm}(t)\}$, $\{\hat{h}_{lm}(t)\}$ and $\{\tilde{H}_{lm}(s,t)$ satisfying (8) implies equivalence of P_1 and P_2. Then, it follows that the eigenvalues λ_i, $i =$

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* The uniqueness of $H(s,t)$ follows from positive-definiteness of $R_i(s,t)$, $i = 1,2$, and square integrability of $H(s,t)$.

† Continuity of $(\partial^{2r}/\partial s^r \partial t^r)\, R_i(s,t)$, $i = 1, 2$, and existence of $x^{(r)}(t)$ for almost all $x(t)$ may be replaced by a simpler but stronger condition that $(\partial^{2r+2}/\partial s^{r+1}\partial t^{r+1})$ $R_i(s,t)$, $i = 1, 2$, exist.

‡ From the communication theoretical point of view, singularity corresponds to the case of "perfect reception" where error probability vanishes. For the mathematical definition, see Ref. 7.

0, 1, 2, $\cdots$, exist.[8,3] Next, we explicitly obtain λ_i from (8) and show that $0<\prod_{i=0}^{\infty} \lambda_i<\infty$. Thus, the threshold c of (3) is well defined. Lastly, we prove that

$$\frac{dP_2}{dP_1}(x) = \left(\prod_{i=0}^{\infty} \lambda_i\right)^{-\frac{1}{2}} \exp\left[\tfrac{1}{2}Q(x)\right] \tag{9}$$

for almost all $x(t)$ of both signals. Then, by substituting (9) into the above decision scheme and taking the logarithm of both sides, the assertion is immediately proved.

III. EXTENSION TO TWO NONZERO-MEAN GAUSSIAN SIGNALS

The preceding result can be extended to the case where the means of the two Gaussian signals are no longer zero.* Let P_{11} and P_{22} be two probability measures corresponding to two Gaussian signals with means $m_1(t)$, $m_2(t)$, $0 \leqq t \leqq 1$, and covariances $R_1(s,t)$, $R_2(s,t)$. $m_1(t)$ and $m_2(t)$ are assumed square-integrable while the assumptions on $R_1(s,t)$ and $R_2(s,t)$ remain the same. Introduce a third measure P_{21} corresponding to a Gaussian signal with mean $m_2(t)$ and covariance $R_1(s,t)$. Then, P_{11} and P_{22} are equivalent and

$$\frac{dP_{22}}{dP_{11}}(x) = \frac{dP_{22}}{dP_{21}}(x)\,\frac{dP_{21}}{dP_{11}}(x)$$

for almost all $x(t)$ of all three signals, if and only if P_{22} is equivalent to P_{21}, which in turn is equivalent to P_{11}. According to a previous result,[9] if there exist finite sequences of real numbers $\{\hat{a}_{jl}\}$, and $\{\hat{t}_j\}$, $0 \leqq \hat{t}_j \leqq 1$, and square-integrable functions $\{\tilde{g}_l\}$ which satisfy

$$\sum_{l=0}^{r}\left[\sum_{j=1}^{n_3} \hat{a}_{jl}\,\frac{\partial^l}{\partial s^l} R_1(s,t)\big|_{s=\hat{t}_j} + \int_0^1 \frac{\partial^l}{\partial s^l} R_1(s,t)\tilde{g}_l(s)\,ds\right] = m_2(t) - m_1(t), \qquad 0 \leqq t \leqq 1, \tag{10}$$

for almost all $x(t)$ of the two signals, then P_{11} and P_{21} are equivalent and $(dP_{21}/dP_{11})(x) = \exp[L(x)]$ for almost all $x(t)$ of the two signals, where

$$L(x) = \sum_{l=0}^{r}\left\{\sum_{j=1}^{n_3} \hat{a}_{jl}\,\frac{d^l}{dt^l}\left[x(t) - \frac{m_1(t)+m_2(t)}{2}\right]_{t=\hat{t}_j} + \int_0^1 \frac{d^l}{dt^l}\left[x(t) - \frac{m_1(t)+m_2(t)}{2}\right]\tilde{g}_l(t)\,dt\right\}. \tag{11†}$$

* This extension follows the development in Ref. 3, pp. 1628–1629 and pp. 1636–1637.

† This is the "sure signals-in-noise" counterpart of the result in Section II, namely, the generalized optimum receiver of two sure signals in Gaussian noise.

The remaining half of showing the equivalence of P_{21} and P_{22} and obtaining $(dP_{22}/dP_{21})(x)$ is accomplished simply by replacing $x(t)$ with $x(t) - m_2(t)$ in the result in Section II. Thus, upon combination, we conclude that, if there exist a set of finite sequences $\{\hat{a}_{jl}\}$, $\{\hat{t}_j\}$, $\{\tilde{g}_l(t)\}$ satisfying (10) and another set of sequences $\{a_{jklm}\}$, $\{s_j\}$, $\{t_j\}$, $\{h_{jlm}(t)\}$, $\{\bar{h}_{jlm}(t)\}$, $\{\hat{h}_{lm}(t)\}$, $\{\tilde{H}_{lm}(s,t)\}$ satisfying (8), then the optimum decision scheme for this case is specified as follows:

$$\text{choose} \quad m_1(t),\, R_1(s,t) \quad \text{if} \quad 2L(x) + Q(x - m_2) < c,$$

$$\text{choose} \quad m_2(t),\, R_2(s,t) \quad \text{otherwise.}$$

IV. EXTENSION TO M GAUSSIAN SIGNALS IN NOISE

The above result can be further extended to the problem of discriminating among M Gaussian signals in Gaussian noise.* Let $m_i(t)$, $R_i(s,t)$ and α_i, $i = 1, 2, \cdots, M$, be the means, covariances and *a priori* probabilities of the signals, and $R_o(s,t)$ the noise covariance where the noise mean is assumed zero. The assumptions concerning $m_i(t)$, $R_i(s,t)$ and $R_o(s,t)$ are the same as in Section III.† Denote by P_{ii} the probability measure corresponding to the ith signal plus the noise, and by P_o the measure corresponding to the noise alone. Then, according to the theory of the generalized maximum likelihood test,[11] if each P_{ii} is equivalent to P_o,‡ then the optimum decision is to choose that $m_i(t)$ and $R_i(s,t)$ for which $\alpha_i(dP_{ii}/dP_o)(x)$ is maximum as a function of i.§ Observe that, if the ith signal plus the noise and the noise alone are interpreted as the two Gaussian signals of Section III with means $m_i(t)$ and zero, and covariances $R_o(s,t) + R_i(s,t)$ and $R_o(s,t)$, then the condition for equivalence of P_{ii} and P_o and the expression for $(dP_{ii}/dP_o)(x)$ are obtained simply by the following changes: $m_1(t) \to 0$, $m_2(t) \to m_i(t)$, $R_1(s,t) \to R_o(s,t)$, $R_2(s,t) \to R_o(s,t) + R_i(s,t)$. Thus, we conclude that, if for each i there exist a set of finite sequences $\{\hat{a}_{ijl}\}$, $\{\hat{t}_{ij}\}$ and $\{\tilde{g}_{il}(t)\}$ satisfying

$$\sum_{l=0}^{r} \left[\sum_{j=1}^{n_{il}} \hat{a}_{ijl} \frac{\partial^l}{\partial s^l} R_o(s,t)|_{s=t_{ij}} + \int_0^1 \frac{\partial^l}{\partial s^l} R_0(s,t) \tilde{g}_{il}(s)\, ds \right] = m_i(t),$$

$$0 \leqq t \leqq 1,$$

* This extension follows the development in Ref. 10, pp. 2192–2194.

† $R_i(s,t)$ need not be strictly positive-definite.

‡ Equivalence of P_{ii} and P_o corresponds to the condition that the ith Gaussian signal cannot be detected perfectly in the presence of this noise.

§ If $\alpha_i(dP_{ii}/dP_o)(x)$ becomes maximum at more than one value of i, choose the lowest of such i-values. See Ref. 11.

and another set of finite sequences $\{a_{ijklm}\}$, $\{s_{ij}\}$, $\{t_{ij}\}$, $\{h_{ijlm}(t)\}$, $\{\tilde{h}_{ijlm}(t)\}$, $\{\hat{h}_{ilm}(t)\}$ and $\{\tilde{H}_{ilm}(s,t)\}$ satisfying

$$\sum_{l,m=0}^{r} \left\{ \sum_{j,k=1}^{n_{i1}} a_{ijklm} \frac{\partial^l}{\partial t^l} R_o(s,t)|_{t=s_{ij}} \frac{\partial^m}{\partial s^m} [R_o(s,t) + R_i(s,t)]_{s=s_{ik}} \right.$$

$$+ \sum_{j=1}^{n_{i2}} \int_0^1 \left[\frac{\partial^l}{\partial t^l} R_o(s,t)|_{t=t_{ij}} h_{ijlm}(u) \frac{\partial^m}{\partial u^m} (R_o(u,t) + R_i(u,t)) \right.$$

$$\left. + \frac{\partial^l}{\partial u^l} R_o(s,u)\tilde{h}_{ijlm}(u) \frac{\partial^m}{\partial s^m} (R_o(s,t) + R_i(s,t))|_{s=t_{ij}} \right] du$$

$$+ \int_0^1 \frac{\partial^l}{\partial u^l} R_o(s,u)\hat{h}_{ilm}(u)$$

$$\cdot \frac{\partial^m}{\partial u^m} [R_o(s,t) + R_i(s,t)]\, du + \int_0^1 \int_0^1 \frac{\partial^l}{\partial u^l} R_o(s,u)\tilde{H}_{ilm}(u,v)$$

$$\left. \cdot \frac{\partial^m}{\partial v^m} [R_o(v,t) + R_i(v,t)]\, du\, dv \right\} = R_i(s,t), \qquad 0 \leqq s,t \leqq 1,$$

then the optimum decision is to choose that signal $(m_i(t), R_i(s,t))$ for which $2L_i(x) + Q_i(x - m_i) + c_i$ is maximum as a function of i, where $L_i(x)$ and $Q_i(x)$ are defined by (11) and (6) with $\hat{a}_{jl}$, $\hat{t}_j$, $\tilde{g}_l(t)$, and a_{jklm}, s_j, t_j, $h_{jlm}(t)$, $\tilde{h}_{jlm}(t)$ $\hat{h}_{lm}(t)$, $\tilde{H}_{lm}(s,t)$ replaced by $\hat{a}_{ijl}$, $\hat{t}_{ij}$, $\tilde{g}_{il}(t)$, and a_{ijklm}, s_{ij}, t_{ij}, $h_{ijlm}(t)$, $\tilde{h}_{ijlm}(t)$ $\hat{h}_{ilm}(t)$, $\tilde{H}_{ilm}(s,t)$, respectively, and

$$c_i = 2 \log \alpha_i - \sum_{n=0}^{\infty} \log \lambda_n^{(i)},$$

where $\lambda_n^{(i)}$, $n = 0, 1, 2, \cdots$, are the eigenvalues of the extension of $I + R_o^{-\frac{1}{2}} R_i R_o^{-\frac{1}{2}}$ to the whole of $\mathfrak{L}_2$.

APPENDIX

Let P_1 and P_2 be two Gaussian measures associated with a separable and measurable process $\{x(t),\ 0 \leqq t \leqq 1\}$ with means zero and covariances $R_1(s,t)$ and $R_2(s,t)$.

Theorem: Suppose $R_1(s,t)$ *and* $R_2(s,t)$ *are (strictly) positive-definite,* $(\partial^{2r}/\partial s^r \partial t^r)R_1(s,t)$ *and* $(\partial^{2r}/\partial s^r \partial t^r)R_2(s,t)$, $0 \leqq r < \infty$, *exist and are continuous on* $[0,1] \times [0,1]$, *and almost all sample functions have the* r*th derivatives with respect to* P_1 *and* P_2. *If there exist a set of finite sequences** $\{a_{iklm}\}$ $\{s_j\}$, $\{t_j\}$, $\{h_{jlm}(t)\}$, $\{\tilde{h}_{jlm}(t)\}$ $\{\hat{h}_{lm}(t)\}$ *and* $\{\tilde{H}_{lm}(s,t)\}$ *which satisfy* (8), *then*

* The definitions of these sequences are given after (5).

(*i*) P_1 *and* P_2 *are equivalent, i.e.,* $P_1 \equiv P_2$,
(*ii*) (10) *holds a.s.,* $[P_1 , P_2]$.*

Proof: For simplicity, we introduce the following notations:

$$R_{i_s l}(u,t) = \frac{\partial^l}{\partial s^l} R_i(s,t)\big|_{s=u} , \qquad R_{i_t m}(s,v) = \frac{\partial^m}{\partial t^m} R_i(s,t)\big|_{t=v} ,$$

$$R_{i_s l_t m}(u,v) = \frac{\partial^{l+m}}{\partial s^l \, \partial t^m} R_i(s,t)\big|_{s=u,t=v} , \qquad i = 1, 2,$$

$$K_1(s,t) = \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} a_{jklm} R_{1_t l}(s,s_j) R_{2_s m}(s_k , t),$$

$$K_2(s,t) = \sum_{l,m=0}^{r} \sum_{j=1}^{n_2} [R_{1_t l}(s,t_j)(R_{2_t m} h_{jlm})(t) + (R_{1_t l}\tilde{h}_{jlm})(s) R_{2_s m}(t_j , t)],$$

$$K_3(s,t) = \sum_{l,m=0}^{r} \int_0^1 R_{1_t l}(s,u) \hat{h}_{lm}(u) R_{2_s m}(u,t) \, du,$$

$$K_4(s,t) = \sum_{l,m=0}^{r} \int_0^1 \int_0^1 R_{1_t l}(s,u) \tilde{H}_{lm}(u,v) R_{2_s m}(v,t) \, du \, dv.$$

Note $K_i(s,t)$, $i = 1, 2, 3, 4$, are square-integrable. Again, we delete the arguments s and t of the kernels to denote the corresponding integral operators. Thus, (8) becomes

$$\sum_{i=1}^{4} K_i = R_2 - R_1 , \tag{12}$$

hence,

$$R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}} - I = \sum_{i=1}^{4} R_1^{-\frac{1}{2}} K_i R_1^{-\frac{1}{2}}. \tag{13}$$

(*i*) To establish $P_1 \equiv P_2$, it suffices to prove that $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$ is densely defined on $\mathfrak{L}_2$ and $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}} - I$ is of Hilbert-Schmidt type, i.e., $\|R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}} - I\| < \infty$.[8,2,3]† The principal tool to be used for this proof is the following expansion:[12]

$$R_{1_s l_t m}(s,t) = \sum_i \mu_i f_i^{(l)}(s) f_i^{(m)}(t), \qquad 0 \leqq l, m \leqq r, \tag{14}$$

uniformly on $[0,1] \times [0,1]$, where $\mu_i > 0$ and $f_i(t)$, $i = 0, 1, 2 \cdots$, are the eigenvalues and orthonormalized eigenfunctions of R_1 .

To prove that $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$ is densely defined on $\mathfrak{L}_2$, it sufficies to show

* "a.s. $[P_1 , P_2]$" is the abbreviation of "almost surely with respect to P_1 and P_2".
† $\|A\|$ denotes the Hilbert-Schmidt norm of an operator A.

that $R_1^{-\frac{1}{2}}R_2$ is bounded since $R_1^{-\frac{1}{2}}$ is densely defined. Now, by applying the formula $\|A\|^2 = \operatorname{tr} A^*A = \sum_i (f_i , A^*Af_i)$ to the individual terms of $R_1^{-\frac{1}{2}}K_1$ first, we obtain

$$\| R_1^{-\frac{1}{2}}K_1 \| \leqq \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} | a_{jklm} |$$
$$\cdot \left[\sum_i \left| \int_0^1 (R_1^{-\frac{1}{2}}f_i)(u) R_{1t^l}(u,s_j) \right|^2 \int_0^1 R_{2s^m}(s_k , u) R_{2t^m}(u,s_k)\, du \right]^{\frac{1}{2}}$$
$$= \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} | a_{jklm} | \left| \sum_i \mu_i \, | f_i^{(l)}(s_j) |^2 R^2_{2s^mt^m}(s_k , s_k) \right|^{\frac{1}{2}}$$
$$= \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} | a_{jklm} | \, | R_{1s^lt^l}(s_j , s_j) R^2_{2s^mt^m}(s_k , s_k) |^{\frac{1}{2}},$$

where (14) is used for the last two equalities. Similarly,

$$\| R_1^{-\frac{1}{2}}K_2 \| \leqq \sum_{l,m=0}^{r} \sum_{j=1}^{n_2} \{ [R_{1s^lt^l}(t_j , t_j)(h_{jlm} , R^2_{2s^mt^m}h_{jlm})]^{\frac{1}{2}}$$
$$+ [(h_{jml} , R_{1s^lt^l}\tilde{h}_{jlm}) R^2_{2s^mt^m}(t_j , t_j)]^{\frac{1}{2}} \},$$

$$\| R_1^{-\frac{1}{2}}K_3 \| \leqq \sum_{l,m=0}^{r} | \operatorname{tr} (\hat{R}_{1s^lt^l,m} R^2_{2s^mt^m}) |^{\frac{1}{2}},$$

$$\| R_1^{-\frac{1}{2}}K_4 \| \leqq \sum_{l,m=0}^{r} | \operatorname{tr} (\tilde{R}_{1s^lt^l,m} R^2_{2s^mt^m}) |^{\frac{1}{2}},$$

where $R_{1s^lt^l,m}(s,t) = \hat{h}_{lm}(s) R_{1s^lt^l}(s,t) \hat{h}_{lm}(t)$, $\tilde{R}_{1s^lt^l,m} = \tilde{H}_{ml} R_{1s^lt^l} \tilde{H}_{lm}$. Hence, from (13), $\|R_1^{-\frac{1}{2}}R_2\| < \infty$.

To prove $\|R_1^{-\frac{1}{2}}R_2R_1^{-\frac{1}{2}} - I\| < \infty$, we apply the formula $\|A\|^2 = \sum \|Af_i\|^2$ to the individual terms of $R_1^{-\frac{1}{2}}K_1R_1^{-\frac{1}{2}}$ first. Thus, we obtain

$$\| R_1^{-\frac{1}{2}}K_1R_1^{-\frac{1}{2}} \| \leqq \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} | a_{jklm} | \left| \sum_i \| R_1^{-\frac{1}{2}}R_{2t^m}(\cdot,s_k) \mu_i^{\frac{1}{2}} f_i^{(l)}(s_j) \|^2 \right|^{\frac{1}{2}}$$
$$= \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} | a_{jklm} | \, | R_{1s^lt^l}(s_j , s_j) |^{\frac{1}{2}} \, \| R_1^{-\frac{1}{2}}R_{2t^m}(\cdot,s_k) \|,$$

where (14) is used twice, and $R_1^{-\frac{1}{2}}R_{2t^m}(\cdot,s_k)$ denotes the result of $R_1^{-\frac{1}{2}}$ acting on an s-function $R_{2t^m}(s,s_k)$. By differentiating both sides of (8) with respect to t, we obtain

$$R_{2t^m}(s,s_k) = R_{1t^m}(s,s_k) + \sum_{l,m=0}^{r} a_{jklm} R_{1t^l}(s,s_j) R_{s^mt^m}(s_k , s_k)$$
$$+ \sum_{l,m=0}^{r} \sum_{j=1}^{n_2} [R_{1t^l}(s,t_j)(R_{2s^mt^m}h_{jlm})(s_k) + (R_{1t^l}\tilde{h}_{jlm})(s) R_{2s^mt^m}(t_j , s_k)]$$

$$+ \sum_{l,m=0}^{r} \int_0^1 R_{1t^l}(s,u)\hat{h}_{lm}(u)R_{2s^m t^m}(u,s_k)\, du$$

$$+ \sum_{l,m=0}^{r} \int_0^1 \int_0^1 R_{1t^l}(s,u)\tilde{H}_{lm}(u,v)R_{2s^m t^m}(v,s_k)\, du\, dv. \tag{15}$$

Thus,

$$\| R_1^{-\frac{1}{2}} R_{2t^m}(\cdot\,,s_k) \| \leqq | R_{1s^m t^m}(s_k\,,\,s_k) |^{\frac{1}{2}}$$

$$+ \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} | a_{jklm} | \, | R_{1s^l}(s_j\,,\,s_j) |^{\frac{1}{2}} R_{2s^m t^m}(s_k\,,\,s_k)$$

$$+ \sum_{l,m=0}^{r} \sum_{j=1}^{n_2} \{ | R_{1s^l t^l}(t_j\,,\,t_j) |^{\frac{1}{2}} \, | (R_{2s^m t^m} h_{jlm})(s_k) |$$

$$+ (\tilde{h}_{jlm}\,,\,R_{1s^l t^l} h_{jlm})^{\frac{1}{2}} \, | R_{2s^m t^m}(s_j\,,\,s_k) | \}$$

$$+ \sum_{l,m=0}^{r} (R_{2s^m t^m}(s_k\,,\,\cdot),\, \hat{R}_{1s^l t^l,m}\, R_{2s^m t^m}(\cdot\,,s_k))^{\frac{1}{2}}$$

$$+ \sum_{l,m=0}^{r} (R_{2s^m t^m}(s_k\,,\,\cdot),\, \tilde{R}_{1s^l t^l,m}\, R_{2s^m t^m}(\cdot\,,s_k))^{\frac{1}{2}}$$

$$< \infty. \tag{16}$$

Hence,

$$\| R_1^{-\frac{1}{2}} K_1 R_1^{-\frac{1}{2}} \| < \infty.$$

Similarly,

$$\| R_1^{-\frac{1}{2}} K_2 R_1^{-\frac{1}{2}} \| \leqq \sum_{l,m=0}^{r} \sum_{j=1}^{n_2} [| R_{1s^l t^l}(t_j\,,\,t_j) |^{\frac{1}{2}} \, \| R_1^{-\frac{1}{2}} R_{2t^m} h_{jlm} \|$$

$$+ (\tilde{h}_{jlm}\,,\,R_{1s^l t^l} \tilde{h}_{jlm})^{\frac{1}{2}} \, \| R_1^{-\frac{1}{2}} R_{2t^m}(\cdot\,,t_j) \|].$$

From (15),

$$\| R_1^{-\frac{1}{2}} R_{2t^m} h_{jlm} \| \leqq (h_{jlm}\,,\,R_{1s^m t^m} h_{jlm})^{\frac{1}{2}}$$

$$+ \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} | a_{jklm} | \, | R_{1s^l t^l}(s_j\,,\,s_j) |^{\frac{1}{2}} \, | (R_{2s^m t^m} h_{jlm})(s_k) |$$

$$+ \sum_{l,m=0}^{r} \sum_{j=1}^{n_2} [| R_{1s^l t^l}(t_j\,,\,t_j) |^{\frac{1}{2}} \, | (h_{jlm}\,,\,R_{2s^m t^m} h_{jlm}) |$$

$$+ (\tilde{h}_{jlm}\,,\,R_{1s^l t^l} \tilde{h}_{jlm})^{\frac{1}{2}} \, | (R_{2s^m t^m} h_{jlm})(t_j) |]$$

$$+ \sum_{l,m=0}^{r} (h_{jlm}\,,\,R_{2s^m t^m} \hat{R}_{1s^l t^l,m}\, R_{2s^m t^m} h_{jlm})^{\frac{1}{2}}$$

$$+ \sum_{l,m=0}^{r} (h_{jlm}\,,\,R_{2s^m t^m} \tilde{R}_{1s^l t^l,m}\, R_{2s^m t^m} h_{jlm})^{\frac{1}{2}}$$

$$< \infty,$$

and, from (16)

$$\| R_1^{-\frac{1}{2}} R_{2ll}(\cdot, t_j) \| < \infty .$$

Hence,

$$\| R_1^{-\frac{1}{2}} K_2 R_1^{-\frac{1}{2}} \| < \infty .$$

Similarly,

$$\begin{aligned}
&\| R_1^{-\frac{1}{2}} K_3 R_1^{-\frac{1}{2}} \| \\
&\leqq \sum_{l,m=0}^{r} \left[\sum_i \mu_i \left\| R_1^{-\frac{1}{2}} \int_0^1 R_{2tm}(\cdot, u) \hat{h}_{lm}(u) f_i^{(l)}(u)\, du \right\|^2 \right]^{\frac{1}{2}} \\
&\leqq \sum_{l,m=0}^{r} \left\{ | \operatorname{tr} (\hat{R}_{1_s l_t l',m} R_{1_s m_t m}) |^{\frac{1}{2}} + \sum_{l',m'=0}^{r} \sum_{j,k=1}^{n_1} | a_{jkl'm'} | \right. \\
&\cdot | R_{1_s l'_t l'}(s_j, s_j)(R_{2_s m'_t m}(s_k, \cdot), \hat{R}_{1_s l_t l,m} R_{2_s m_t m'}(\cdot, s_k) |^{\frac{1}{2}} \\
&+ \sum_{l',m'=0}^{r} \sum_{j=1}^{n_2} [| R_{1_s l'_t l'}(t_j, t_j)(R_{2_s m_t m'} h_{jl'm'}, \hat{R}_{1_s l_t l,m} R_{2_s m_t m'} h_{jl'm'}) |^{\frac{1}{2}} \\
&+ |(\bar{h}_{jl'm'}, R_{1_s l'_t l'} \bar{h}_{jl'm'})(R_{2_s m'_t m}(t_j, \cdot), \hat{R}_{1_s l_t l,m} R_{2_s m_t m'}(\cdot, t_j))|^{\frac{1}{2}}] \\
&+ \sum_{l',m'=0}^{r} | \operatorname{tr} (\hat{R}_{1_s l'_t l',m'} R_{2_s m'_t m} \hat{R}_{1_s l_t l,m} R_{2_s m_t m'}) |^{\frac{1}{2}} \\
&\left. + \sum_{l',m'=0}^{r} | \operatorname{tr} (\tilde{R}_{1_s l'_t l',m'} R_{2_s m'_t m} \hat{R}_{1_s l_t l,m} R_{2_s m_t m'}) |^{\frac{1}{2}} \right\} \\
&< \infty .
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\| R_1^{-\frac{1}{2}} K_4 R_1^{-\frac{1}{2}} \| \\
&\leqq \sum_{l,m=0}^{r} [\sum_i \mu_i \| R_1^{-\frac{1}{2}} R_{2tm} \tilde{H}_{ml} f_i^{(l)} \|^2]^{\frac{1}{2}} \\
&\leqq \sum_{l,m=0}^{r} \left\{ | \operatorname{tr} (\tilde{R}_{1_s l_t l,m} R_{1_s m_t m}) |^{\frac{1}{2}} + \sum_{l',m'=0}^{r} \sum_{j,k=1}^{n_1} | a_{jkl'm'} | \right. \\
&\cdot | R_{1_s l'_t l'}(s_j, s_j)(R_{2_s m'_t m}(s_k, \cdot), \tilde{R}_{1_s l_t l,m} R_{2_s m_t m'}(\cdot, s_k)) |^{\frac{1}{2}} \\
&+ \sum_{l',m'=0}^{r} \sum_{j=1}^{n_2} [| R_{1_s l'_t l'}(t_j, t_j)(h_{jl'm'}, R_{2_s m'_t m} \tilde{R}_{1_s l_t l,m} R_{2_s m_t m'} h_{jl'm'}) |^{\frac{1}{2}} \\
&+ | (\bar{h}_{jl'm'}, \ddot{R}_{1_s l'_t l'} \bar{h}_{jl'm'})(R_{2_s m'_t m}(t_j, \cdot), \tilde{R}_{1_s l_t l,m} R_{2_s m_t m'}(\cdot, t_j)) |^{\frac{1}{2}}] \\
&+ \sum_{l',m'=0}^{r} | \operatorname{tr} (\hat{R}_{1_s l'_t l',m'} R_{2_s m'_t m} \tilde{R}_{1_s l_t l,m} R_{2_s m_t m'}) |^{\frac{1}{2}}
\end{aligned}$$

$$+ \sum_{l',m'=0}^{r} | \operatorname{tr} (\tilde{R}_{1sl'tl',m'} R_{2sm'tm} \tilde{R}_{1sltl,m} R_{2smtm'}) |^{\frac{1}{2}} \Big\}$$

$$< \infty .$$

Therefore, from (13),

$$|| R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}} - I || < \infty .$$

(*ii*) We have established in (*i*) that $R_1^{-\frac{1}{2}} R_2 R_1^{-\frac{1}{2}}$ is bounded and densely defined on $\mathfrak{L}_2$. Hence, it has a unique extension to the whole of $\mathfrak{L}_2$, which we denote by M. Since $M - I$ is a Hilbert-Schmidt operator, M has eigenvalues and orthonormal eigenfunctions, which we denote by λ_i and $\varphi_i(t)$, $i = 0, 1, 2, \cdots$. Note $0 < \lambda_i \leqq | M |$, where $| M |$ is the norm of M. Then[13]

$$R_{1sltm}(s,t) = \sum_i (R_1^{\frac{1}{2}}\varphi_i)^{(l)}(s)(R_1^{\frac{1}{2}}\varphi_i)^{(m)}(t).$$

$$0 \leqq l, m \leqq r, \qquad (17)$$

$$R_{2sltm}(s,t) = \sum_i \lambda_i (R_1^{\frac{1}{2}}\varphi_i)^{(l)}(s)(R_1^{\frac{1}{2}}\varphi_i)^{(m)}(t),$$

uniformly on $[0,1] \times [0,1]$.

Let $\{\varphi_{in}\}$ be sequences of functions in the domain of $R_1^{-\frac{1}{2}}$ such that $\varphi_i = \text{l.i.m. } \varphi_{in}$ for each i. Multiply both sides of (12) by $(R_1^{-\frac{1}{2}}\varphi_{in})(s)$ and $(R_1^{-\frac{1}{2}}\varphi_{in})(t)$, integrate with respect to s and t, and let $n \to \infty$. Then, the four terms on the left-hand side become

$$(R_1^{-\frac{1}{2}}\varphi_{in} , K_1 R_1^{-\frac{1}{2}}\varphi_{in})$$

$$= \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} a_{jklm} (R_1^{-\frac{1}{2}}\varphi_{in} , R_{1tl}(\cdot ,s_j))(R_{2sm}(s_k , \cdot), R_1^{-\frac{1}{2}}\varphi_{in})$$

$$= \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} a_{jklm} \sum_{\nu} (\varphi_{in} , \varphi_\nu)(R_1^{\frac{1}{2}}\varphi_\nu)^{(l)}(s_j) \sum_{\nu} \lambda_\nu (R_1^{\frac{1}{2}}\varphi_\nu)^{(m)}(s_k)(\varphi_\nu , \varphi_{in}),$$

where (17) is used for the second equality. By virtue of (17) again, we can define an s-function $R_{1tl}^{\frac{1}{2}}(s,u)$ for any $u \in [0,1]$ by

$$R_{1tl}^{\frac{1}{2}}(s,u) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=0}^{n} \varphi_\nu(s)(R_1^{\frac{1}{2}}\varphi_\nu)^{(l)}(u).$$

Then

$$\sum_{\nu} (\varphi_{in} , \varphi_\nu)(R_1^{\frac{1}{2}}\varphi_\nu)^{(l)}(s_j) = (\varphi_{in} , R_{1tl}^{\frac{1}{2}}(\cdot ,s_j)),$$

$$\sum_{\nu} \lambda_\nu (R_1^{\frac{1}{2}}\varphi_\nu)^{(m)}(s_k)(\varphi_\nu , \varphi_{in}) = (R_{1sm}^{\frac{1}{2}}(s_k , \cdot), M\varphi_{in}),$$

and

$$\lim_{n\to\infty} (\varphi_{in}, R_{1tl}^{\frac{1}{2}}(\cdot, s_j)) = (\varphi_i, R_{1tl}^{\frac{1}{2}}(\cdot, s_j)) = (R_1^{\frac{1}{2}}\varphi_i)^{(l)}(s_j),$$

$$\lim_{n\to\infty} (R_{1sm}^{\frac{1}{2}}(s_k, \cdot), M\varphi_{in}) = (R_{1sm}^{\frac{1}{2}}(s_k, \cdot), M\varphi_i) = \lambda_i(R_1^{\frac{1}{2}}\varphi_i)^{(m)}(s_k).$$

Hence,

$$\lim_{n\to\infty} (R_1^{-\frac{1}{2}}\varphi_{in}, K_1R_1^{-\frac{1}{2}}\varphi_{in}) = \lambda_i \sum_{l,m=0}^{r} \sum_{j,k=1}^{n_1} a_{jklm}(R_1^{\frac{1}{2}}\varphi_i)^{(l)}(s_j)(R_1^{\frac{1}{2}}\varphi_i)^{(m)}(s_k).$$

Similarly,

$$\begin{aligned}\lim_{n\to\infty} &(R_1^{-\frac{1}{2}}\varphi_{in}, K_2R_1^{-\frac{1}{2}}\varphi_{in})\\ &= \lim_{n\to\infty} \sum_{l,m=0}^{r} \sum_{j=1}^{n_2} [(R_1^{-\frac{1}{2}}\varphi_{in}, R_{1tl}(\cdot, t_j))(R_{2tm}h_{jlm}, R_1^{-\frac{1}{2}}\varphi_{in})\\ &\quad + (R_1^{-\frac{1}{2}}\varphi_{in}, R_{1tl}\bar{h}_{jlm})(R_{2sm}(t_j, \cdot), R_1^{-\frac{1}{2}}\varphi_{in})]\\ &= \lambda_i \sum_{l,m=0}^{r} \sum_{j=1}^{n_2} (R_1^{\frac{1}{2}}\varphi_i)^{(l)}(t_j)((R_1^{\frac{1}{2}}\varphi_i)^{(m)}, h_{jlm} + \bar{h}_{jlm}),\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} &(R_1^{-\frac{1}{2}}\varphi_{in}, K_3R_1^{-\frac{1}{2}}\varphi_{in})\\ &= \lim_{n\to\infty} \sum_{l,m=0}^{r} \int_0^1 (R_1^{-\frac{1}{2}}\varphi_{in}, R_{1tl}(\cdot, u))\hat{h}_{lm}(u)(R_{2sm}(u, \cdot), R_1^{-\frac{1}{2}}\varphi_{in})\, du\\ &= \lambda_i \sum_{l,m=0}^{r} \int_0^1 (R_1^{\frac{1}{2}}\varphi_i)^{(l)}(u)\hat{h}_{lm}(u)(R_1^{\frac{1}{2}}\varphi_i)^{(m)}(u)\, du,\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} &(R_1^{-\frac{1}{2}}\varphi_{in}, K_4R_1^{-\frac{1}{2}}\varphi_{in})\\ &= \lim_{n\to\infty} \sum_{l,m=0}^{r} \int_0^1 \int_0^1 (R_1^{-\frac{1}{2}}\varphi_{in}, R_{1tl}(\cdot, u))\tilde{H}_{lm}(u,v)(R_{2sm}(v, \cdot), R_1^{-\frac{1}{2}}\varphi_{in})\, du\, dv\\ &= \lambda_i \sum_{l,m=0}^{r} ((R_1^{\frac{1}{2}}\varphi_i)^{(l)}, \tilde{H}_{lm}(R_1^{\frac{1}{2}}\varphi_i)^{(m)}).\end{aligned}$$

On the other hand, the right-hand side becomes

$$\lim_{n\to\infty} (R_1^{-\frac{1}{2}}\varphi_{in}, (R_2 - R_1)R_1^{-\frac{1}{2}}\varphi_{in}) = \lim_{n\to\infty} (\varphi_{in}, (M - I)\varphi_{in}) = \lambda_i - 1.$$

Hence, by equating both sides and dividing by λ_i,

$$1 - \frac{1}{\lambda_i} = \sum_{l,m=0}^{r} \Bigg[\sum_{j,k=1}^{n_1} a_{jklm}(R_1^{\frac{1}{2}}\varphi_i)^{(l)}(s_j)(R_1^{\frac{1}{2}}\varphi_i)^{(m)}(s_k)$$

$$+ \sum_{j=1}^{n_2} (R_1^{\frac{1}{2}}\varphi_i)^{(l)}(t_j)((R_1^{\frac{1}{2}}\varphi_i)^{(m)}, h_{jlm} + \tilde{h}_{jlm})$$

$$+ \int_0^1 (R_1^{\frac{1}{2}}\varphi_i)^{(l)}(u)\hat{h}_{lm}(u)(R_1^{\frac{1}{2}}\varphi_i)^{(m)}(u)\,du + ((R_1^{\frac{1}{2}}\varphi_i)^{(l)}, \tilde{H}_{lm}(R_1^{\frac{1}{2}}\varphi_i)^{(m)})\Bigg]. \quad (18)$$

Thus,

$$\sum_i \left(1 - \frac{1}{\lambda_i}\right) = \sum_{l,m=0}^{r} \Bigg[\sum_{j,k=1}^{n_1} a_{jklm} R_{1_{\bullet}l_{l}m}(s_j\ , s_k)$$

$$+ \sum_{j=1}^{n_2} R_{1_{\bullet}l_{l}m}(h_{jlm} + \tilde{h}_{jlm})(t_j) + \int_0^1 R_{1_{\bullet}l_{l}m}(u,u)\hat{h}_{lm}(u)\,du + \operatorname{tr}(R_{1_{\bullet}l_{l}m}\tilde{H}_{ml})\Bigg]$$

$$< \infty,$$

where (17) is used repeatedly. Hence,*

$$0 < \prod_{i=0}^{\infty} \lambda_i < \infty.$$

$$\frac{dP_2}{dP_1}(x) = \left(\prod_{i=0}^{\infty} \lambda_i\right)^{-\frac{1}{2}} \exp\left[\tfrac{1}{2} \sum_i \left(1 - \frac{1}{\lambda_i}\right)\eta_i^2(x)\right], \quad \text{a.s., } [P_1],$$

where

$$\eta_i(x) = \underset{n\to\infty}{\text{l.i.m.}}\ (x, R_1^{-\frac{1}{2}}\varphi_{in}), \qquad [P_1\ , P_2] \qquad i = 0, 1, 2, \cdots . \qquad (19)$$

Now, $x^{(l)}(t)$ has the following orthogonal expansion:[13]

$$x^{(l)}(t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{i=0}^{n} \eta_i(x)(R_1^{\frac{1}{2}}\varphi_i)^{(l)}(t), \qquad [P_1], \qquad 0 \leqq l \leqq r,$$

uniformly in t. Hence, there exists a subsequence of the partial sums $\sum_{i=0}^{n_r} \eta_i(x)(R_1^{\frac{1}{2}}\varphi_i)^{(l)}(t)$ which converges a.s. $[P_1]$ to $x^{(l)}(t)$, uniformly in t. Therefore, from (18) and (19),

$$\sum_i \left(1 - \frac{1}{\lambda_i}\right)\eta_1^2(x)$$

$$= \lim_{n_r\to\infty} \sum_{i=0}^{n_r} \left(1 - \frac{1}{\lambda_i}\right)\eta_i^2(x)$$

$$= \sum_{l,m=0}^{r} \Bigg[\sum_{j,k=1}^{n_1} a_{jklm} x^{(l)}(s_j) x^{(m)}(s_k) + \sum_{j=1}^{n_2} x^{(l)}(t_j)(x^{(m)}, h_{jlm} + \tilde{h}_{jlm})$$

$$+ \int_0^1 x^{(l)}(u)\hat{h}_{lm}(u)x^{(m)}(u)\,du + (x^{(l)}, \tilde{H}_{lm}x^{(m)})\Bigg], \quad \text{a.s.} \quad [P_1],$$

which completes the proof of (*ii*).

* See Ref. 3, pp. 1653–1654.

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